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Surjections between Euclidean spaces, changing variable formula and Brouwer fixed point theorem

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1. Differentiation for vector functions

The Euclidean norm of a vector $x \in \mathbb{R}^m$ is

$$|x| = \sqrt{(x^1)^2 + \cdots + (x^m)^2}.$$

Let $U \subset \mathbb{R}^m$, $f : U \to \mathbb{R}^n$, $a \in U^\circ$. If there is $n \times m$ matrix A s.t.

$$\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{a}) - \boldsymbol{A}\boldsymbol{h} = \boldsymbol{o}(|\boldsymbol{h}|), \qquad \textbf{as } |\boldsymbol{h}| \to \mathbf{0}, \tag{1}$$

we call A the derivative of f at a, write $A = f'(a)$ or $A = Df(a)$.

Let $f = \left(f^1, \ldots, f^n\right)$, denote the i^{th}-row of A by A^i, the i component of (1) is

$$f^i(a+h) - f^i(a) - A^i h = o(|h|).$$

Thus f^i is differentiable at a, and

$$A^i = \nabla f^i(a) = \left(\partial_1 f^i, \ldots, \partial_m f^i\right).$$

Therefore

$$A = \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_m f^1 \\ \vdots & & \vdots \\ \partial_1 f^n & \cdots & \partial_m f^n \end{pmatrix}, \qquad \text{the Jacobian matrix of } f \text{ at } a.$$

When $m = n$, the Jacobian determinant of A is

$$J_f(a) = \frac{\partial(f^1, \ldots, f^m)}{\partial(x^1, \ldots, x^m)} = \det \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_m f^1 \\ \vdots & & \vdots \\ \partial_1 f^n & \cdots & \partial_m f^n \end{pmatrix}. \tag{2}$$

Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$. $f : U \to \mathbb{R}^n$, $g : V \to \mathbb{R}^\ell$. If $f(U) \subset V$, we have

$$g \circ f : U \to \mathbb{R}^\ell.$$

Thm 1 (Chain role). If f is differentiable at $a \in U$, g is differentiable at $b = f(a)$, then $g \circ f$ is differentiable at a and $\boldsymbol{(g \circ f)'(a) = g'(b) f'(a)}$.

If $y = g(u)$, $u = f(x)$, then

$$\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial y^\ell}{\partial x^1} & \cdots & \frac{\partial y^\ell}{\partial x^m} \end{pmatrix}_a = \begin{pmatrix} \frac{\partial y^1}{\partial u^1} & \cdots & \frac{\partial y^1}{\partial u^n} \\ \vdots & & \vdots \\ \frac{\partial y^\ell}{\partial u^1} & \cdots & \frac{\partial y^\ell}{\partial u^n} \end{pmatrix}_b \begin{pmatrix} \frac{\partial u^1}{\partial x^1} & \cdots & \frac{\partial u^1}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial u^n}{\partial x^1} & \cdots & \frac{\partial u^n}{\partial x^m} \end{pmatrix}_a, \quad \frac{\partial y^i}{\partial x^j} = \sum_{k=1}^{n} \frac{\partial y^i}{\partial u^k} \frac{\partial u^k}{\partial x^j}.$$

Thm 2 (Inverse function). Let Ω be open in $\mathbb{R}^m$, $f:\Omega\to\mathbb{R}^m$ be C^1, $a\in\Omega$, $b=f(a)$. If $\det f'(a)\neq 0$, there are $U\in\mathscr{U}_a$ and $V\in\mathscr{U}_b$, s.t. $f:U\to V$ is diffeomorphism. (illustrating the basic idea of differential calculus)

Rek 1. $\det f'(a)\neq 0$ means $f'(a):\mathbb{R}^m\to\mathbb{R}^m$ is a linear isomorphism; then f is locally invertible.

Cor 1 (Local surjection). Let Ω be open in $\mathbb{R}^m$, $f:\Omega\to\mathbb{R}^n$ be C^1, $a\in\Omega$. If $\operatorname{rank} f'(a)=n$, then $b\in[f(\Omega)]^\circ$

Rek 2. $\operatorname{rank} f'(a)=n$ means $f'(a):\mathbb{R}^m\to\mathbb{R}^n$ is linear surjection; then f is locally surjective.

Pf(Cor1). .

$$f'(a)=\begin{pmatrix}\partial_1 f^1 & \cdots & \partial_n f^1 & \cdots & \partial_m f^1\\ \vdots & & \vdots & & \vdots\\ \partial_1 f^n & \cdots & \partial_n f^n & \cdots & \partial_m f^n\end{pmatrix},\quad \det\begin{pmatrix}\partial_1 f^1 & \cdots & \partial_n f^1\\ \vdots & & \vdots\\ \partial_1 f^n & \cdots & \partial_n f^n\end{pmatrix}\neq 0.$$

Let $F:\Omega\to\mathbb{R}^m$, $F(x)=(f^1(x),\dots,f^n(x),x^{n+1},\dots,x^m)$. then

$$F'(a)=\begin{pmatrix}\left(\partial_i f^j\right)_{i,j=1,\dots,n} & \left(\partial_i f^j\right)_{i>n}\\ 0 & I_{m-n}\end{pmatrix}$$

is invertible. We apply the Inverse Function Theorem to F.

2. Suejectivity of $f : \mathbb{R}^m \to \mathbb{R}^n$ and FTA

Thm 3 (FTA). Let $a_i \in \mathbb{C}$, $n \geq 1$,

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

be a polynomial, then $\exists \xi \in \mathbb{C}$ s.t. $p(\xi) = 0$.

- Gauss gave the first proof more than 300 years ago. Even in 21st century, new proofs keep emerging Sen (2000); Lazer & Leckband (2010).
- Usual proofs use Complex Analysis or Algebraic Topology, many proofs are collected in Fine & Rosenberger (1997).
- A proof via Green theorem can be found in the textbook Ouyang et al. (2003).
- Lazer & Leckband (2010) applied Fourier inverse transform to prove FTA.
- Sen (2000) applied Inverse Fun Thm, the proof need topological concepts such as open sets (and connectivity) in subspace of $\mathbb{C}$.
- Our **motivation** was to avoid subspace topology.

Let $z = x + iy$, $p(z) = u(x,y) + iv(x,y)$. View p as a map $p : \mathbb{R}^2 \to \mathbb{R}^2$,

$$p(x,y) = (u(x,y), v(x,y)).$$

From Cauchy-Riemann equation we know

$$p'(z) = 0 \iff \det p'(x,y) = 0, \quad (x,y) \text{ is critical point of } p : \mathbb{R}^2 \to \mathbb{R}^2$$

Since $p'(z)$ is polynomial of order $(n-1)$, the map $p : \mathbb{R}^2 \to \mathbb{R}^2$ has finitely many critical points. The fact

$$\lim_{|(x,y)|\to\infty} |p(x,y)| = +\infty,$$

brings our attention to a classical result (advanced calculus exercise):

Pro 1 ((Deimling, 1985, Page 24)). If the C^1-map $f : \mathbb{R}^n \to \mathbb{R}^n$ is coercive:

$$\lim_{|x|\to\infty} |f(x)| = +\infty, \tag{3}$$

and $\det Df(x) \neq 0$ for $\forall x \in \mathbb{R}^n$, then $f(\mathbb{R}^n) = \mathbb{R}^n$.

Rek 3. (1) If we can weaken $\det Df(x) \neq 0$ for $\forall x \in \mathbb{R}^n$ to $\det Df(x) \neq 0$ except finitely many x, then FTA follows immediately.

(2) (3) means that $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$. This motivates our Thm 5.

Def 1. Let $f:\mathbb{R}^m\to\mathbb{R}^n$ be C^1, $a\in\mathbb{R}^m$. If $Df(a):\mathbb{R}^m\to\mathbb{R}^n$ is not surjective (i.e., $\operatorname{rank} Df(a)<n$), we call a a critical point of f.

Thm 4 (Local surjection). Let $\Omega\subset\mathbb{R}^m$, $f:\Omega\to\mathbb{R}^n$ be C^1. If $a\in\Omega^\circ$ is not critical (i.e. $\operatorname{rank} Df(a)=n$), then $f(a)\in[f(\Omega)]^\circ$.

Thm 5 (Liu & Liu (2018)). If the C^1-map $f:\mathbb{R}^m\to\mathbb{R}^n$ has only finitely many critical points, $n\geq 2$, $f(\mathbb{R}^m)$ is closed in $\mathbb{R}^n$, then $f(\mathbb{R}^m)=\mathbb{R}^n$.
. (necessary condition)

Pf. Let K be the critical set of f, then $f(K)$ is also finite.

- $\mathbb{R}^m\backslash K$ is open in $\mathbb{R}^m$. By Thm 4, $\forall x\in\mathbb{R}^m\backslash K$, $f(x)$ is interior to $A=f(\mathbb{R}^m\backslash K)$. hence A is open in $\mathbb{R}^n$.
- By assumption,
$$A\cup f(K)=f(\mathbb{R}^m\backslash K)\cup f(K)=f(\mathbb{R}^m)$$
is **closed**.
- Noting that $f(\mathbb{R}^m)=\overline{f(\mathbb{R}^m)}\supset\overline{A}$, it suffices to prove the intuitive result:

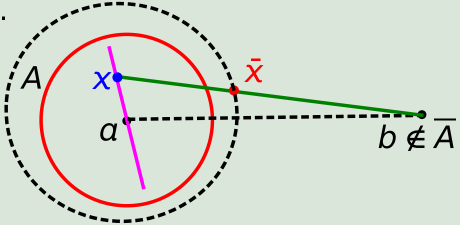
if the union of open set A and a finite set is closed, then $\overline{A}=\mathbb{R}^n$.

Lem 1. Let $n \geq 2$, A be nonempty open set in $\mathbb{R}^n$. If there are k points $p_i \in \mathbb{R}^n$ s.t. $A \cup \{p_i\}_{i=1}^k$ is closed, then $\overline{A} = \mathbb{R}^n$.

Pf. Since A is open, $A \cap \partial A = \emptyset$.

$$A \cup \{p_i\} = \overline{A \cup \{p_i\}} = A \cup \partial A \cup \{p_i\} \implies \partial A \subset \{p_i\},$$

So ∂A is finite.

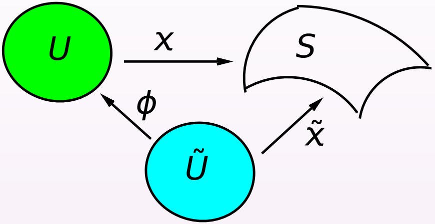


Cor 2. Assume M is m-dimensional smooth manifold without boundary, C^1-map $f : M \to \mathbb{R}^n$ has only finitely many critical points, $n \geq 2$. If $f(M)$ is closed in $\mathbb{R}^n$, then $f(M) = \mathbb{R}^n$.

Cor 3. Let $n \geq 2$, M be m-dimensional compact manifold without boundary. If $f : M \to \mathbb{R}^n$ is C^1-map, then f has infinitely many critical points.

3. Changing variable for multiple integrals

Motivated by do Carmo (1976) (CVF for double integral via Green Theorem), we assume CVF for $(m-1)$-integrals, define **surface integral** in $\mathbb{R}^m$ and prove the **Divergence Theorem**, then prove CVF for m-integrals.



3.1. Surface integral and Divergence Theorem

- Let U be Jordan measurable closed domain in $\mathbb{R}^{m-1}$, a C^1-parametrized surface is a C^1-map $x : U \to \mathbb{R}^m$ satisfying $\operatorname{rank}\left(\partial x^i/\partial u^j\right) = m-1$ and injective in U°.
- x is equivalent with $\tilde{x} : \tilde{U} \to \mathbb{R}^m$ if there is diffism $\phi : \tilde{U} \to U$ s.t. $\tilde{x} = x \circ \phi$.

  Then $\tilde{x}(\tilde{U}) = x(U)$, we can identify the equivalent class $[x]$ with $x(U)$, called **smooth surface**, and denoted by $S = x(U)$ or $S = [x : U \to \mathbb{R}^m]$.

Using Cramer we know that a **normal vector** of $S=[x:U\to\mathbb{R}^m]$ at $x(u)$ is

$$N(u)=\left(\frac{\partial(x^2,\dots,x^m)}{\partial(u^1,\dots,u^{m-1})},\dots,(-1)^{m+1}\frac{\partial(x^1,\dots,x^{m-1})}{\partial(u^1,\dots,u^{m-1})}\right).$$

The **surface integral** of cont $f:S\to\mathbb{R}$ on $S=[x:U\to\mathbb{R}^m]$ is defined by

$$\int_S f(x)\mathrm{d}\sigma=\int_U f(x(u))\,|N(u)|\,\mathrm{d}u. \tag{4}$$

By $(m-1)$-dim CVF, the RHS is **indept** with the parametrization of S.

For piece-wise smooth surface $\Sigma=\bigcup_{i=1}^{\ell}S_i$, where $S_i=x_i(U_i)$ interiorly-disjoint, set

$$\int_\Sigma f\mathrm{d}\sigma=\sum_{i=1}^{\ell}\int_{S_i}f\mathrm{d}\sigma. \qquad x_i(U_i^\circ)\cap x_j(U_j^\circ)=\varnothing.$$

Thm 6 (Divergence Theorem). Let $D\subset\mathbb{R}^m$ be bounded closed domain, ∂D piece-wise smooth, $F\in C^1(D,\mathbb{R}^m)$, n is unit outer normal of ∂D, then

$$\int_D \operatorname{div}F\,\mathrm{d}x=\int_{\partial D}F\cdot n\,\mathrm{d}\sigma. \qquad \int_D\partial_i f\,\mathrm{d}x=\int_{\partial D}fn^i\,\mathrm{d}\sigma \quad \text{for } f\in C^1(D).$$

3.2. Simple domain

A bdd domain Ω is **simple**, if there is $(m-1)$-dim C^1–parametrized surface $x: U \to \mathbb{R}^m$ s.t. $\partial\Omega = x(U)$. Note that U is closed, x is injective in U°.

Exm 1. Balls B in $\mathbb{R}^m$ is simple. For $m=3$, we take

$$x: [0,\pi]\times[0,2\pi] \to \mathbb{R}^3, \quad (\varphi,\theta) \mapsto (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

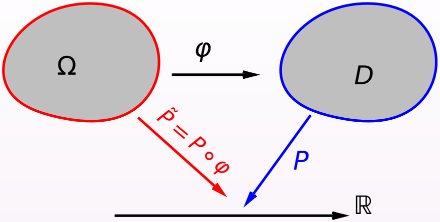
Thm 7 (Liu & Zhang (2017)). Let D and Ω be bdd open domain in $\mathbb{R}^m$ with C^1-boundary, Ω simple, the C^2-map $\varphi: \overline{\Omega} \to \overline{D}$ maps $\partial\Omega$ onto ∂D diffeomly, $f \in C(\overline{D})$, then

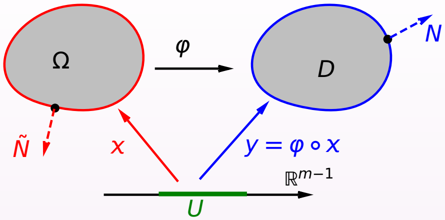
$$\int_D f(y)\mathrm{d}y = \pm \int_\Omega f(\varphi(x)) J_\varphi(x)\mathrm{d}x, \quad \text{where } J_\varphi(x) = \det \varphi'(x).$$

Rek 4. (1) Using mollifier, we may assume that f is the restriction to $\overline{D}$ of smooth fun on $\mathbb{R}^m$. Thus we can take $P \in C^1(\mathbb{R}^m)$ s.t. $\partial P/\partial y^1 = f$.
(2) φ only good on bdry, this leads to Brouwer Fixed Point Theorem (Thm9).
(3) See also Lax (1999, 2001); Taylor (2002); Ivanov (2005).

Lax 99: $f \in C_0(\mathbb{R}^m)$
φ is identity outside some ball

$$\int_{\mathbb{R}^m} f(y)\mathrm{d}y = \int_{\mathbb{R}^m} f(\varphi(x)) \det\left(\frac{\partial y}{\partial x}\right)\mathrm{d}x.$$





Pf. Let $\varphi : y^i = y^i(x^1, \dots, x^m)$. Take $P \in C^1(\mathbb{R}^m)$ s.t. $\partial P/\partial y^1 = f$, set $\tilde{P} = P \circ \varphi$.
Let $x : U \to \mathbb{R}^m$ be parametrization of $\partial\Omega$, then $y = \varphi \circ x$ is parametrion of ∂D,

$$N = \left(\frac{\partial(y^2, \dots, y^m)}{\partial(u^1, \dots, u^{m-1})}, \dots, (-1)^{m+1} \frac{\partial(y^1, \dots, y^{m-1})}{\partial(u^1, \dots, u^{m-1})} \right)$$

is normal vector of ∂D, $n = \pm N/|N| = (n^1, \dots, n^m)$ unit normal.

Let $A = (A_1, \dots, A_m)$, $\tilde{N} = (\tilde{N}^1, \dots, \tilde{N}^m)$, where

$$A_i = (-1)^{i+1} \frac{\partial(y^2, \dots, y^m)}{\partial(x^1, \dots, \hat{x}^i, \dots, x^m)}, \qquad \tilde{N}^i = (-1)^{i+1} \frac{\partial(x^1, \dots, \hat{x}^i, \dots, x^m)}{\partial(u^1, \dots, u^{m-1})}.$$

then $\tilde{n} = \pm\tilde{N}/|\tilde{N}|$ is unit normal vec of $\partial\Omega$.

By Cauchy-Binet formula,

$$\pm n^1|N| = \frac{\partial(y^2,\dots,y^m)}{\partial(u^1,\dots,u^{m-1})}$$
$$= \sum_{i=1}^{m} \frac{\partial(y^2,\dots,y^m)}{\partial(x^1,\dots,\hat{x}^i,\dots x^m)} \frac{\partial(x^1,\dots,\hat{x}^i,\dots x^m)}{\partial(u^1,\dots,u^{m-1})} = A\cdot\tilde{N}.$$

By Divergence Theorem, $P\circ y = P\circ(\varphi\circ x) = (P\circ\varphi)\circ x = \tilde{P}\circ x$

$$\int_D f(y)\mathrm{d}y = \int_D \frac{\partial P}{\partial y^1}\mathrm{d}y = \int_{\partial D} Pn^1 \mathrm{d}\sigma$$
$$= \pm\int_U P(y(u))n^1(u)\,|N(u)|\,\mathrm{d}u \qquad \tilde{N} = \pm|\tilde{N}|\tilde{n}$$
$$= \pm\int_U \tilde{P}(x(u))(A\cdot\tilde{N})\,\mathrm{d}u = \pm\int_U (\tilde{P}A\cdot\tilde{n})|\tilde{N}|\,\mathrm{d}u$$
$$= \pm\int_{\partial\Omega} \tilde{P}A\cdot\tilde{n}\,\mathrm{d}\sigma = \pm\int_{\Omega} \mathrm{div}(\tilde{P}A)\,\mathrm{d}x. \tag{5}$$

To compute $\operatorname{div}(\tilde{P}A)$, let $y_i^k = \partial y^k/\partial x^i$, then A_i is algebraic cofactor of y_i^1 in

$$J_\varphi(x) = \frac{\partial(y^1,\dots,y^m)}{\partial(x^1,\dots,x^m)} = \det\begin{pmatrix} y_1^1 & y_2^1 & \cdots & y_i^1 & \cdots & y_m^1 \\ y_1^2 & y_2^2 & \cdots & y_i^2 & \cdots & y_m^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1^m & y_2^m & \cdots & y_i^m & \cdots & y_m^m \end{pmatrix}.$$

By Hadamard identity we get $\operatorname{div} A = 0$, so

$$\sum_{i=1}^m y_i^j A_i = \delta_1^j \frac{\partial(y^1,\dots,y^m)}{\partial(x^1,\dots,x^m)} = \delta_1^j J_\varphi(x), \qquad \tilde{P}(x) = P(\varphi(x))$$

$$\begin{aligned}\operatorname{div}(\tilde{P}A) &= \nabla\tilde{P}\cdot A + \tilde{P}\operatorname{div} A = \nabla\tilde{P}\cdot A = \sum_{i=1}^m \frac{\partial\tilde{P}}{\partial x^i}A_i = \sum_{i=1}^m \Big(\sum_{j=1}^m \frac{\partial P}{\partial y^j}\frac{\partial y^j}{\partial x^i}\Big)A_i \\ &= \sum_{j=1}^m \frac{\partial P}{\partial y^j}\Big(\sum_{i=1}^m y_i^j A_i\Big) = (\partial_{y^1}P)J_\varphi(x) = f(\varphi(x))J_\varphi(x).\end{aligned}$$

From (5) we get

$$\int_D f(y)\mathrm{d}y = \pm\int_\Omega f(\varphi(x))J_\varphi(x)\mathrm{d}x.$$

Cor 4. Under assumptions of Thm 7 , if J_φ does not change sign on $\overline{\Omega}$, then

$$\int_D f(y)\mathrm{d}y = \int_\Omega f(\varphi(x))\,|J_\varphi(x)|\mathrm{d}x.$$

3.3. General domain

Thm 8. Let D and Ω be Jordan measurable bounded open domains in $\mathbb{R}^m$, $f \in C(\overline{D})$, $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^m)$, $\varphi : \Omega \to D$ is diffeomorphism, then

$$\int_D f(y)\mathrm{d}y = \int_\Omega f(\varphi(x))|J_\varphi(x)|\mathrm{d}x.$$

Pf. Since $f = f_+ - f_-$, $f_\pm = \frac{1}{2}(|f| \pm f) \in C(\overline{D})$, we may assume $f \geq 0$.
Set $\tilde{f}(x) = f(\varphi(x))|J_\varphi(x)|$.

(1) $\forall \varepsilon > 0$, choose disjoint balls $B_i \subset \Omega$ s.t.

$$\int_\Omega \tilde{f}(x)\mathrm{d}x - \varepsilon \leq \sum_i \int_{B_i} \tilde{f}(x)\mathrm{d}x = \sum_i \int_{\varphi(B_i)} f(y)\mathrm{d}y \leq \int_D f(y)\mathrm{d}y.$$

(2) Letting $\varepsilon \to 0$ we get $\int_\Omega \tilde{f}(x)\mathrm{d}x \leq \int_D f(y)\mathrm{d}y$.

(3) Similarly, $\int_D f(y)\mathrm{d}y \leq \int_\Omega \tilde{f}(x)\mathrm{d}x$.

4. Brouwer fixed point theorem (BFPT)

Thm 9 (Brouwer). Let B be the unit closed ball in $\mathbb{R}^m$, $g : B \to B$ be continuous. Then g has a fixed point.

It is well known that to prove Thm 9 it suffices to prove

Pro 2. There does not exist $\varphi \in C^2(B, \mathbb{R}^m)$ s.t. $\varphi(B) \subset \partial B$ and $\varphi|_{\partial B} = 1_{\partial B}$.

Pf(Motivated by Báez-Duarte (1993)). Take

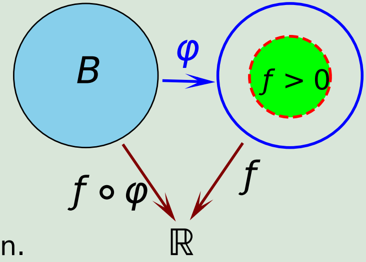
$$f(y) = \begin{cases} \sqrt{1 - 4\,|y|^2}, & |y| \le \frac{1}{2}, \\ 0, & \frac{1}{2} < |y| \le 1. \end{cases}$$

Then $f(\varphi(x)) = 0$ for $x \in B$.

View $\varphi : x \mapsto y$ as transformation, Thm 7 yields

$$0 < \int_B f(y) \mathrm{d}y$$

$$= \pm \int_B f(\varphi(x)) \det \left(\frac{\partial y}{\partial x} \right) \mathrm{d}x = 0, \text{ a contradiction.}$$



Rek 5. ✳ Most people learn the proof of BFPT for the first time as application of homology theory in Algebraic Topology.

✳ Proved as application of Stokes formula on Differentiable manifolds.

✳ Elementary proofs can also be found in Milnor (1978); Rogers (1980); Kannai (1981).

Rek 6. Advantages of our proof of Changing variable formula:

(1) It is just clever computation (Cauchy-Binet, etc), more easy to follow.

(2) Theory of surface integral (including Divergence Theorem) is developed during the proof.

(3) As by product we get Brouwer fixed point theorem.

Exm 2 (Application of BFPT). Let A be invertible, $f : \mathbb{R}^n \to \mathbb{R}^n$ verify

$$\lim_{|x|\to\infty} \frac{|f(x)|}{|x|} = 0,$$

$$\sum_{j=1}^{n} a_i^j x^j = f^i(x^1, \dots, x^n)$$

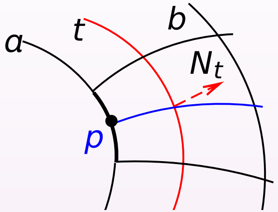
$$\begin{cases} -\Delta u = \lambda u + f(u), \\ u|_{\partial\Omega} = 0. \end{cases}$$

$$\lim_{|t|\to\infty} \frac{f(t)}{t} = 0.$$

Then the nonlinear equation $Ax = f(x)$ is solvable ($Ax = b$, Cramer rule).

5. Coarea formula and applications

Influenced by Mei (2020), I present coarea formula.



Thm 10. Let $G \subset \mathbb{R}^m$ be bdd open, $f \in C^2(G)$, $\nabla f(x) \neq 0$ for $\forall x \in G$. $\Omega = f^{-1}[a,b] \subset G$.

If $g \in C(\Omega)$, then $\displaystyle\int_\Omega g = \int_a^b \mathrm{d}t \int_{f^{-1}(t)} \frac{g}{|\nabla f|} \mathrm{d}\sigma.$

.

Pf(Motivated by (Wu et al., 1989, §11)). For $p \in f^{-1}(a)$, let $x(\cdot, p)$ be solution of

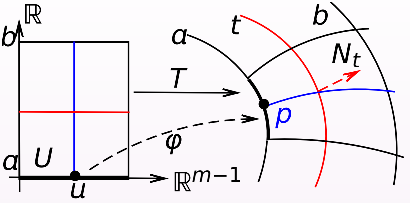
$$x' = \frac{\nabla f(x)}{|\nabla f(x)|^2}, \quad x(a) = p. \tag{6}$$

Then $x(b,p) \in f^{-1}(b)$. (ODE is applied in Calculus)

Let $\varphi : U \to \mathbb{R}^m$ be par of $f^{-1}(a)$, then the C^1-map $T : U \times [a,b] \to \mathbb{R}^m$,

$$T(u,t) = x(t-a, \varphi(u)) \quad \text{is interiorly injective,}$$

$T(\cdot, t) : U \to \mathbb{R}^m$ is par of $f^{-1}(t)$, with normal $N_t(u)$.



Expanding $\det T'(u,t)$ and using (6) yield

$$\left|\det T'(u,t)\right| = \frac{|N_t(u)|}{|\nabla f(T(u,t))|} \neq 0.$$

So T is diffeomorphism on $U^\circ \times (a,b)$.
By changing variable and Fubini

$$\int_\Omega g(x)\mathrm{d}x = \int_{T(U\times(a,b))} g(x)\mathrm{d}x \qquad x = T(u,t)$$

$$= \int_{U\times(a,b)} g(T(u,t)) \left|\det T'(u,t)\right| \mathrm{d}u\mathrm{d}t$$

$$= \int_a^b \mathrm{d}t \int_U \frac{g(T(u,t))}{|\nabla f(T(u,t))|} |N_t(u)| \,\mathrm{d}u$$

$$= \int_a^b \mathrm{d}t \int_{f^{-1}(t)} \frac{g}{|\nabla f|} \mathrm{d}\sigma.$$

Exm 3. Let $g \in C^1(B_R \times [0,R])$, then for $r \in (0,R)$,

$$\frac{d}{dr}\int_{B_r} g(x,r)\,dx = \int_{B_r} \frac{\partial}{\partial r} g(x,r)\,dx + \int_{\partial B_r} g(x,r)\,d\sigma.$$

Pf. By coarea formula,

$$\int_{B_r} g(x,r)\,dx = \int_0^r dt \int_{|x|=t} g(x,r)\,d\sigma.$$

Applying

$$\frac{d}{dr}\int_0^r F(r,t)\,dt = F(r,r) + \int_0^r \partial_r F(r,t)\,dt$$

to

$$F(r,t) = \int_{|x|=t} g(x,r)\,d\sigma,$$

we deduce

$$\begin{aligned}\frac{d}{dr}\int_{B_r} g(x,r)\,dx &= \int_{|x|=r} g(x,r)\,d\sigma + \int_0^r dt \int_{|x|=t} \partial_r g(x,r)\,d\sigma \\ &= \int_{\partial B_r} g(x,r)\,d\sigma + \int_{B_r} \partial_r g(x,r)\,dx.\end{aligned}$$

Exm 4. Let B be unit ball in $\mathbb{R}^m$, $f \in C^1(B)$, $f|_{\partial B} = 0$. Find

$$I = \lim_{\varepsilon \to 0^+} \int_{B \setminus B_\varepsilon} \frac{x \cdot \nabla f(x)}{|x|^m} \mathrm{d}x, \qquad \text{where } B_\varepsilon : |x| \le \varepsilon.$$

Pf. By defn of surface integrals (4), $\int_{|x|=t} g(x)\,\mathrm{d}\sigma = t^{m-1} \int_{|y|=1} g(ty)\,\mathrm{d}\sigma$.

$$\int_{B \setminus B_\varepsilon} \frac{x \cdot \nabla f(x)}{|x|^m} \mathrm{d}x = \int_\varepsilon^1 \mathrm{d}t \int_{|x|=t} \frac{x \cdot \nabla f(x)}{|x|^m} \mathrm{d}\sigma$$

$$= \int_\varepsilon^1 \left(t^{m-1} \int_{|y|=1} \frac{(ty) \cdot \nabla f(ty)}{|ty|^m} \mathrm{d}\sigma \right) \mathrm{d}t$$

$$= \int_\varepsilon^1 \mathrm{d}t \int_{|y|=1} \nabla f(ty) \cdot y\,\mathrm{d}\sigma = \int_{|y|=1} \mathrm{d}\sigma \int_\varepsilon^1 \frac{d}{dt} f(ty)\,\mathrm{d}t$$

$$= \int_{|y|=1} [-f(\varepsilon y)]\,\mathrm{d}\sigma \to -f(0)\omega_m.$$

Rek 7. This can also be solved using divergence theorem.

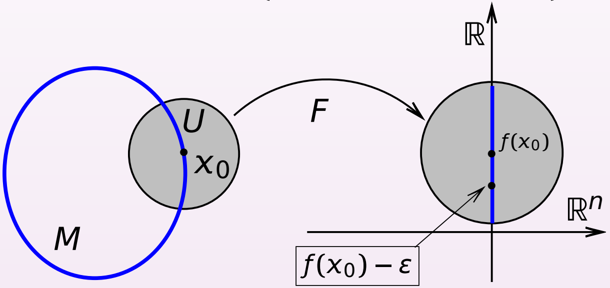
Pf of L-multipliers, $F(x) = (f(x), g^1(x), \ldots, g^n(x))$. By Thm4

$$\min\{f : \mathbb{R}^m \to \mathbb{R}\} \qquad F : U \to \mathbb{R}^{n+1}$$

$$g^1(x) = 0$$
$$\vdots$$
$$g^n(x) = 0$$

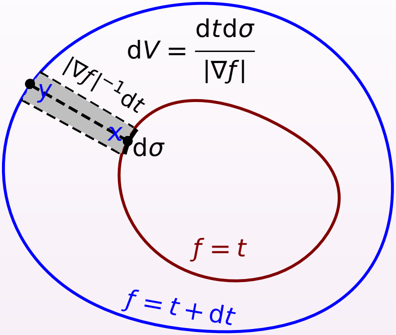
$$\operatorname{rank} F'(x_0) = \operatorname{rank} \begin{pmatrix} \nabla f(x_0) \\ \nabla g^1(x_0) \\ \vdots \\ \nabla g^n(x_0) \end{pmatrix} = n,$$

thus $\nabla f(x_0) \in \operatorname{span}\left\{\nabla g^1(x_0), \ldots, \nabla g^n(x_0)\right\}$.



Coarea and method of element

Let $G \subset \mathbb{R}^m$, $f : G \to \mathbb{R}$. $\Omega = f^{-1}[a,b]$, $g : \Omega \to \mathbb{R}$.



Take surface element $\mathrm{d}\sigma$ at $x \in f^{-1}(t)$. Let y be the intersection of $f^{-1}(t+\mathrm{d}t)$ and normal line of $f^{-1}(t)$ at x. Then

$$\mathrm{d}t = f(y) - f(x) \approx \nabla f(x) \cdot (y - x),$$

$$|y - x| = \frac{\mathrm{d}t}{|\nabla f(x)|}.$$

Volume of the gray column with base $\mathrm{d}\sigma$ and high $|y - x|$ is

$$\mathrm{d}V = \frac{\mathrm{d}t\mathrm{d}\sigma}{|\nabla f(x)|}, \quad \mathrm{d}m = \frac{g(x)}{|\nabla f(x)|}\mathrm{d}t\mathrm{d}\sigma. \quad \int_{f=t} \mathrm{d}m = \text{mass of } f^{-1}[t, t+dt]$$

Hence mass of Ω is

$$\int_\Omega g(x)\mathrm{d}x = \int_a^b \mathrm{d}t \int_{f=t} \frac{g(x)}{|\nabla f(x)|}\mathrm{d}\sigma.$$

Exm 5. Let $g \in C^1(B_R \times [0,R])$, then for $r \in (0,R)$,

$$\frac{d}{dr}\int_{B_r} g(x,r)\,dx = \int_{B_r} \frac{\partial}{\partial r} g(x,r)\,dx + \int_{\partial B_r} g(x,r)\,d\sigma.$$

Pf. By coarea formula,

$$\int_{B_r} g(x,r)\,dx = \int_0^r dt \int_{|x|=t} g(x,r)\,d\sigma.$$

Applying

$$\frac{d}{dr}\int_0^r F(r,t)\,dt = F(r,r) + \int_0^r \partial_r F(r,t)\,dt$$

to

$$F(r,t) = \int_{|x|=t} g(x,r)\,d\sigma,$$

we deduce

$$\begin{aligned}\frac{d}{dr}\int_{B_r} g(x,r)\,dx &= \int_{|x|=r} g(x,r)\,d\sigma + \int_0^r dt \int_{|x|=t} \partial_r g(x,r)\,d\sigma \\ &= \int_{\partial B_r} g(x,r)\,d\sigma + \int_{B_r} \partial_r g(x,r)\,dx.\end{aligned}$$

Chain rule and Cauchy-Binet

$$\begin{pmatrix} \frac{\partial y^2}{\partial u^1} & \cdots & \frac{\partial y^2}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial y^m}{\partial u^1} & \cdots & \frac{\partial y^m}{\partial u^{m-1}} \end{pmatrix} = \begin{pmatrix} \frac{\partial y^2}{\partial x^1} & \cdots & \frac{\partial y^2}{\partial x^i} & \cdots & \frac{\partial y^2}{\partial x^m} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial y^m}{\partial x^1} & \cdots & \frac{\partial y^m}{\partial x^i} & \cdots & \frac{\partial y^m}{\partial x^m} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial x^i}{\partial u^1} & \cdots & \frac{\partial x^i}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial x^m}{\partial u^1} & \cdots & \frac{\partial x^m}{\partial u^{m-1}} \end{pmatrix}$$

$(m-1)\times(m-1)$ $(m-1)\times m$ $m\times(m-1)$

Cauchy-Binet yields

$$\frac{\partial(y^2,\ldots,y^m)}{\partial(u^1,\ldots,u^{m-1})} = \sum_{i=1}^{m} \frac{\partial(y^2,\ldots,y^m)}{\partial(x^1,\ldots,\hat{x}^i,\ldots x^m)} \frac{\partial(x^1,\ldots,\hat{x}^i,\ldots x^m)}{\partial(u^1,\ldots,u^{m-1})}$$

References

Sen A (2000). Notes: Fundamental Theorem of Algebra – Yet Another Proof. Amer. Math. Monthly, 107(9) 842–843.

Lazer AC, Leckband M (2010). The fundamental theorem of algebra via the Fourier inversion formula. Amer. Math. Monthly, 117(5) 455–457.

Fine B, Rosenberger G (1997). The fundamental theorem of algebra. Undergraduate Texts in Mathematics. Springer-Verlag, New York.

Ouyang G, Yao Y, Zhou Y (2003). Mathematical Analysis. Fudan Univ. Press.

Deimling K (1985). Nonlinear functional analysis. Springer-Verlag, Berlin.

Liu P, Liu S (2018). On the surjectivity of smooth maps into Euclidean spaces and the fundamental theorem of algebra. Amer. Math. Monthly, 125(10) 941–943.

do Carmo MP (1976). Differential geometry of curves and surfaces. Prentice-Hall Inc., Englewood Cliffs, N.J. Translated from the Portuguese.

Liu S, Zhang Y (2017). On the change of variables formula for multiple integrals. J. Math. Study, 50(3) 268–276.

Lax PD (1999). Change of variables in multiple integrals. Amer. Math. Monthly, 106(6) 497–501.

Lax PD (2001). Change of variables in multiple integrals. II. Amer. Math. Monthly, 108(2) 115–119.

Taylor M (2002). Differential forms and the change of variable formula for multiple integrals. J. Math. Anal. Appl., 268(1) 378–383.

Ivanov NV (2005). A differential forms perspective on the Lax proof of the change of variables formula. Amer. Math. Monthly, 112(9) 799–806.

Báez-Duarte L (1993). Brouwer's fixed-point theorem and a generalization of the formula for change of variables in multiple integrals. J. Math. Anal. Appl., 177(2) 412–414.

Milnor J (1978). Analytic proofs of the "hairy ball theorem" and the Brouwer fixed-point theorem. Amer. Math. Monthly, 85(7) 521–524.

Rogers CA (1980). A less strange version of Milnor's proof of Brouwer's fixed-point theorem. Amer. Math. Monthly, 87(7) 525–527.

Kannai Y (1981). An elementary proof of the no-retraction theorem. Amer. Math. Monthly, 88(4) 264–268.

Mei J (2020). Mathematical Analysis. Mathematical Analysis.

Wu H, Shen C, Yu Y (1989). Riemannian Geometry. Peking Univ. Press.

Thank you!

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