第十六届全国非线性泛函分析会议

鞍点约化下的临界群及变系数椭圆共振问题

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Contents

1. SPR and critical groups

Let u be an isolated critical point of $f \in C^1(X, R)$, $f(u) = c$.

$$C_q(f, u) = H_q(f_c, f_c \backslash u), \qquad C_q(f, \infty) = H_q(X, f_\alpha)$$

are the critical groups of f, see [Cha93,BL97].

Exm-1. (1) u is loc. min. $\Longrightarrow$ $C_q(f, u) = \delta_{q,0}\mathscr{G}$.

(2) u is loc. max. $\Longrightarrow$ $C_q(f, u) = \delta_{q,\dim X}\mathscr{G}$.

(3) u nondegenerate, $\operatorname{ind}(f, u) = \mu$ $\Longrightarrow$ $C_q(f, u) = \delta_{q,\mu}\mathscr{G}$.

Pro-1 (Morse inequality). Let

$$M_q = \sum_{f'(u)=0} \operatorname{rank} C_q(f, u), \qquad \beta_q = \operatorname{rank} C_q(f, \infty).$$

Then $\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1 + t)Q(t)$.

[Cha93] K.-c. Chang, Infinite-dimensional Morse theory and ..., 1993.

[BL97] T. Bartsch, S. Li, Nonlinear Anal., 28(1997) 419–441.

Pro-2. $X = X^- \oplus X^+$, $f \in C^1(X,R)$, $\kappa > 0$, $v \in X^-$, $w_{1,2} \in X^+$:

$$\pm \langle \nabla f(v+w_1) - \nabla f(v+w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 . \quad (E_\pm)$$

Then $\exists \psi : X^- \to X^+$,

(1) if (E_+) then $\varphi(v) \triangleq f(v+\psi(v)) = \min_{w \in X^+} f(v+w)$.

(2) if (E_-) then $\varphi(v) \triangleq f(v+\psi(v)) = \max_{w \in X^+} f(v+w)$.

Moreover, $\varphi \in C^1(X,R)$,

v critical for $\varphi \iff v + \psi(v)$ critical for f.

Rek-1. In most applications $\min\{\dim X^-, \dim X^+\} < \infty$.

Pob-1. What is the relation between the critical groups of f and φ?

Thm-1 ([LL03]). In the setting of Pro2,

(1) in case (E_+) we have $C_q(f,\infty)\cong C_q(\varphi,\infty)$.

(2) in case (E_-) with $\ell=\dim X^+<\infty$, then

$$C_q(f,\infty)\cong C_{q-\ell}(\varphi,\infty).$$

Thm-2 ([Liu07]). In Pro2, (E_+), $C_q(f,v+\psi(v))\cong C_q(\varphi,v)$.

Since ψ is bounded if ∇f is ([Liu08, Lem 2.1]),

$$\nabla\varphi(v)=P_-\nabla f(v+\psi(v)),$$

if $\nabla f=\mathbf{1}_X-K$ with $K:X\to X$ compact, then $\nabla\varphi=\mathbf{1}_{X^-}-Q$.
The L-S index

$$\operatorname{ind}(\nabla f,v+\psi(v))\qquad\text{and}\qquad\operatorname{ind}(\nabla\varphi,v).\qquad\text{make sense}$$

[LL03] S. Liu, S. Li, Commun. Contemp. Math., 5(2003) 761–773.
[Liu07] S. Liu, J. Math. Anal. Appl., 336(2007) 498–505.
[Liu08] S. Liu, Proc. Roy. Soc. Edinburgh Sect. A, 138(2008) 1281–1289.

Cor-1 ([LM85]). E_+ : $\operatorname{ind}(\nabla\varphi, v) = \operatorname{ind}(\nabla f, v+\psi(v))$.

Pf. By Poincaré-Hopf,

$$\operatorname{ind}(\nabla\varphi, v)=\sum_{q=0}^{\infty}(-1)^q \operatorname{rank} C_q(\varphi, v)=\sum_{q=0}^{\infty}(-1)^q \operatorname{rank} C_q(f, v+\psi(v)) = \operatorname{ind}(\nabla f, v+\psi(v)).$$

The dual of Thm2 for the case (E_-) remains open until recently.

Thm-3. In Pro2 with case (E_-), if $\ell = \dim X^+ < \infty$, then

$$C_q(f, \bar v+\psi(\bar v)) \cong C_{q-\ell}(\varphi, \bar v). \tag{1}$$

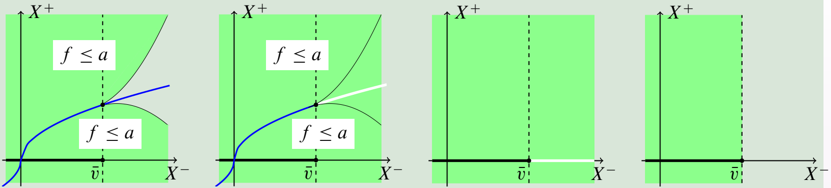
Cor-2. $(E_-) \Rightarrow \operatorname{ind}(\nabla\varphi, v) = (-1)^\ell \operatorname{ind}(\nabla f, v+\psi(v))$.

Rek-2. It follows from (1) that $C_q(f, \bar v+\psi(\bar v)) = 0$ for $q < \ell$.

Pf of Thm3. Assume $\varphi(\bar v) = f(\bar v+\psi(\bar v)) = a$. Note that

$$\varphi(v) = f(v+\psi(v)) = \max_{w\in X^+} f(v+w).$$

[LM85] A. C. Lazer, P. J. McKenna, J. Math. Anal. Appl., 107(1985) 371–395.



if $\varphi(v) \le a$, then for any $w \in X^+$ we have $f(v+w) \le a$. Thus

$$f_a = (\varphi_a \times X^+) \cup \{ (v,w) | f(v+w) \le a, \varphi(v) > a \}.$$

It has been shown in the proof of Thm 1 that

$$f_a \simeq A \triangleq (\varphi_a \times X^+) \cup \{ (v,w) | \varphi(v) > a, w \ne \psi(v) \}$$
$$\simeq (\varphi_a \times X^+) \cup ((X^- \backslash \varphi_a) \times S) \triangleq B,$$

where $S = \{ w \in X^+ | w \ne 0 \}$. Define $H : [0,1] \times B \to B$,

$$H(t,(v,w)) = \begin{cases} (v,w), & \text{if } (v,w) \in \varphi_a \times X^+, \\ ((1-t)v + t\bar{v}, w), & \text{if } (v,w) \in (X^- \backslash \varphi_a) \times S. \end{cases}$$

$H(1,\cdot)$ is a homotopy equivalance between B and $\varphi_a \times X^+$.
We can deform f_a to $\varphi_a \times X^+$, with $(\bar{v}, \psi(\bar{v}))$ to $(\bar{v}, 0)$.

Noting that $(\varphi_a \times X^+)\backslash(\bar{v}, 0) = (\varphi_a \times S) \cup ((\varphi_a \backslash v) \times X^+)$, we have

$$
\begin{aligned}
(f_a, f_a\backslash(\bar{v}, \psi(\bar{v}))) &\simeq (\varphi_a \times X^+, (\varphi_a \times X^+)\backslash(\bar{v}, 0)) \\
&= (\varphi_a \times X^+, (\varphi_a \times S) \cup ((\varphi_a \backslash v) \times X^+)) \\
&= (\varphi_a, \varphi_a \backslash \bar{v}) \times (X^+, S).
\end{aligned}
$$

Passing to homology and apply Künneth,

$$
\begin{aligned}
&C_*(f, \bar{v} + \psi(\bar{v})) = H_*(f_a, f_a\backslash(\bar{v}, \psi(\bar{v}))) \\
&\quad \cong H_*((\varphi_a, \varphi_a\backslash\bar{v}) \times (X^+, S)) \\
&\quad = H_*(\varphi_a, \varphi_a\backslash\bar{v}) \otimes H_*(X^+, S) = H_{*-\ell}(\varphi_a, \varphi_a\backslash\bar{v}) = C_{*-\ell}(\varphi, \bar{v}).
\end{aligned}
$$

Pro-3 ([Liu08, Cor 2.2]). In Pro2, if $\nabla f : X \to X$ is bounded and $\exists K : X \to X$ compact such that $\nabla f = \mathbf{1}_X - K$, then $\nabla \varphi = \mathbf{1}_{(X^-)} - Q$ for some compact $Q : X^- \to X^-$.

[Liu08] S. Liu, Proc. Roy. Soc. Edinburgh Sect. A, 138(2008) 1281–1289.

Thm-4 ([LS01, Thm 2.1]). Let $f \in C^1(X, \mathbb{R})$ satisfy (PS), bounded from below. If $C_\ell(f, \mathbf{0}) \neq 0$ for some $\ell \neq 0$, then f has three critical points. homological 3 Cr.Pts.Thm.

[LS01] J. Liu, J. Su, J. Math. Anal. Appl., 258(2001) 209–222.

2. Applications of SPR

SPR was introduced by [Ama79], and used by many people.
[Cha93,Lon90] used SPR for periodic solutions of HS:

$$-J\dot{z} = H'(t,z)$$

in the case $\left|H''(t,z)\right| \leq C$.
Strongly indefinite functional reduces to finite dim function.

In [Liu09], an approach for computing $C_*(f,\infty)$ via SPR (Thm1), Alexander dual theorem, is developed.

[Ama79] H. Amann, Math. Z., 169(1979) 127–166.
[Cha93] K.-c. Chang, Infinite-dimensional Morse theory and ..., 1993.
[Lon90] Y. M. Long, Sci. China Ser. A, 33(1990) 1409–1419.
[Liu09] S. Liu, Nonlinear Anal., 70(2009) 1965–1974.

2.1. Elliptic BVP: 0 is loc min ($p_0 < \lambda_1$)

Notations $\lim_{|t|\to 0} \frac{p(x,t)}{t} = p_0$, $\lim_{|t|\to\infty} \frac{p(x,t)}{t} = p_\infty$,

$$-\Delta u = p(x,u), \qquad u \in H_0^1(\Omega). \tag{2}$$

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega P(x,u)dx.$$

Thm-5 ([CC94]). $p \in C^1(\mathbb{R})$, $p_0 < \lambda_1$, $p'(t) \le \gamma < \lambda_{m+1}$. If $p_\infty \in (\lambda_m, \lambda_{m+1})$, then (2) has 5 solutions.

[CC94] A. Castro, J. Cossio, SIAM J. Math. Anal., 25(1994) 1554–1561.

Thm-6 ([LZ99]). $p \in C^1(\mathbb{R})$, $p_0 < \lambda_1$, $p'(t) \le \gamma < \lambda_{m+1}$.
If $p_\infty = \lambda_m$,

(p_3) $\exists \alpha \in [0,1)$, $|p(t) - \lambda_m t| \le C(1 + |t|^\alpha)$,

$$\lim_{|t|\to\infty} \frac{1}{|t|^{2\alpha}} \left(P(t) - \frac{1}{2}\lambda_m t^2 \right) = +\infty,$$

then (2) has 5 solutions..

Thm-7 ([Liu07]). $p \in C^1(\mathbb{R})$, $p_0 < \lambda_1$, $p'(t) \le \gamma < \lambda_{m+1}$.
If $p_\infty = \lambda_m$,

$$\lim_{|t|\to\infty} \left(P(t) - \frac{1}{2}\lambda_m t^2 \right) = +\infty,$$

then (2) has 5 solutions.

[LZ99] S. Li, Z. Zhang, Discrete Contin. Dynam. Systems, 5(1999) 489–493.
[Liu07] S. Liu, J. Math. Anal. Appl., 336(2007) 498–505.

2.2. Elliptic BVP: 0 is not loc min ($p_0 > \lambda_1$)

Assume f has a local linking at $\mathbf{0}$.

Thm-8 ([LW98]). $p \in C^1(\Omega \times \mathbb{R})$,

(p_3^-) $\exists \alpha \in [0,1)$, $|p(x,t) - \lambda_m t| \le C\left(1 + |t|^\alpha\right)$. (so $p_\infty = \lambda_m$)

$$\lim_{|t|\to\infty} |t|^{-2\alpha}\left(2P(x,t) - \lambda_m t^2\right) = -\infty,$$

$\partial_t p(x,t) \ge \gamma > \lambda_{m-1}$, then (2) has 3 solutions.

used to control $\operatorname{ind}(f,u)$, and compute $C_*(f,u)$..

Thm-9 ([LTW00]). $p \in C(\Omega \times \mathbb{R})$, $\exists \beta < \lambda_{m+1}$, (2) has 3 sols if

$$\frac{p(x,t) - p(x,s)}{t-s} \le \beta, \quad \lim_{|t|\to\infty} \left(2P(x,t) - \lambda_m t^2\right) = +\infty.$$

[LW98] S. Li, M. Willem, NoDEA, 5(1998) 479–490.
[LTW00] S. Liu, C. Tang, X. Wu, J. Math. Anal. Appl., 249(2000) 289–299.

Thm-10 ([Liu08]). $p \in C(\Omega \times \mathbb{R})$, $\exists \beta > \lambda_{m-1}$, (2) has 3 sols if $\left|\frac{p(x,t)}{t}\right| \le \Lambda$, $\frac{p(x,t)-p(x,s)}{t-s} \ge \beta$, $\lim_{|t|\to\infty} \left(2P(x,t) - \lambda_m t^2\right) = -\infty$.

(1) In Thm 9, since $\dim X^- < \infty$, [LTW00] first observed that although f not (PS), the reduced φ anti-coercive.

(2) In Thm 10,

$$X^- = \bigoplus_{i \ge m} \ker(-\Delta - \lambda_i).$$

Since $\dim X^- = \infty$, it is difficult to prove φ coercive.

To overcome we proved a non vanishing lemma (Lem 2).

(3) Thm 9, Thm 10 rely on the fact that

if f has a local linking at 0, so has φ.

NOT TRUE if p_0 and p_∞ depend on x.

[Liu08] S. Liu, Proc. Roy. Soc. Edinburgh Sect. A, 138(2008) 1281–1289.
[LTW00] S. Liu, C. Tang, X. Wu, J. Math. Anal. Appl., 249(2000) 289–299.

(a) for constant case, decompose at 0 and ∞ W.R.T.

$$-\Delta u = \lambda u, \qquad u \in H_0^1(\Omega).$$

(b) for variable case, at 0 and ∞ decomp W.R.T.

$$-\Delta u = \lambda p_0(x)u \qquad \text{and} \qquad -\Delta u = \lambda p_\infty(x)u.$$

There is a **twist**!

Local linking of f does not descend to φ.

We need Thm 3.

3. Variable coefficients problems

Consider

$$-\Delta u = p(x,u), \qquad u \in H_0^1(\Omega). \tag{3}$$

Assume $p \in C(\Omega \times \mathbb{R}, \mathbb{R})$,

$$|p(x,t)| \le \Lambda |t|. \tag{4}$$

Set $\mathscr{C} = C(\bar{\Omega})$. The for $p \in \mathscr{C}$,

$$-\Delta u = \lambda p(x) u, \qquad u \in H_0^1(\Omega) \tag{5}$$

has eigenvalues $-\infty < \lambda_1(p) < \lambda_2(p) < \cdots$. Assume $\exists p_0 \in \mathscr{C}$,

$$G(x,t) \triangleq P(x,t) - \frac{1}{2} p_0(x) t^2 = o(t^2), \qquad \text{as } |t| \to 0.$$

If $\lambda_k(p_0) = 1$, assume further

$(P_0^{\pm})$ $\exists \delta > 0$, $\pm G(x,t) > 0$ for $0 < |t| \le \delta$. (local linking)

$(P_\infty^{\pm})$ $\exists p_\infty \in \mathscr{C}$, $\lambda_m(p_\infty) = 1$, $\lim\limits_{|t|\to\infty} \left(P(x,t) - \frac{1}{2} p_\infty(x) t^2 \right) = \pm\infty.$

Rek-3. If $\lim_{|t|\to\infty} \dfrac{p(x,t)}{t} = p_\infty(x)$, $\forall i$, $1 \neq \lambda_i(p_\infty)$, no $(P^\infty_\pm)$.

Denote $\lambda_i^0 = \lambda_i(p_0)$, $\lambda_i^\infty = \lambda_i(p_\infty)$,

$$d_n^0 = \sum_{i=1}^n \ker(-\Delta - \lambda_i^0 p_0), \qquad d_n^\infty = \sum_{i=1}^n \ker(-\Delta - \lambda_i^\infty p_\infty).$$

For $a, b \in C(\bar{\Omega})$, $a \preceq b$ if $a \le b$ and $a < b$ on $\tilde{\Omega} \subset \Omega$, $|\tilde{\Omega}| > 0$.

Thm-11. Assume (4), (P^+_∞). If $\exists \beta \preceq \lambda^\infty_{m+1} p_\infty$ s.t.

$$(p(x,t) - p(x,s))(t-s) \le \beta(x)(t-s)^2,$$

then (3) has two nontrivial solutions in one of

(1) (P_0^+), $d_k^0 \neq d_m^\infty$,

(2) (P_0^-), $d_{k-1}^0 \neq d_m^\infty$.

Thm-12. Assume (4), (P^-_∞). If $\exists \beta \succeq \lambda^\infty_{m-1} p_\infty$ s.t.

$$(p(x,t)-p(x,s))(t-s) \geq \beta(x)(t-s)^2,$$

then (3) has two nontrivial solutions in one of

(1) (P^+_0), $d^0_k \neq d^\infty_{m-1}$,

(2) (P^-_0), $d^0_{k-1} \neq d^\infty_{m-1}$.

Rek-4. (1) Thm 11 is easier than Thm 12: the reduced functional is finite dim.

(2) In Thm 11, instead of (4), it suffices to assume subcritical growth.

4. Proof of Thm 12

We find critical points of $f : H_0^1(\Omega) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega P(x,u)dx.$$

Lem-1. $(P_0^+) \Longrightarrow C_{d_k^0}(f, \mathbf{0}) \neq 0$. $(P_0^-) \Longrightarrow C_{d_{k-1}^0}(f, \mathbf{0}) \neq 0$.

Pf. For (P_0^-), set $V_0 = \ker(-\Delta - \lambda_k^0 p_0)$,

$$V_- = \bigoplus_{i<k} \ker(-\Delta - \lambda_i^0 p_0), \qquad V_+ = \overline{\bigoplus_{i>k} \ker(-\Delta - \lambda_i^0 p_0)}.$$

Then $\dim V_- = d_{k-1}^0$. We can show that for $\|u\|$ small

$$f(u) \le 0,\ u \in V_-, \qquad f(u) > 0,\ u \in V_0 \oplus V_+ \setminus \mathbf{0}. \quad \text{(local linking)}$$

The desired result will then follow from [Liu89].

[Liu89] J. Q. Liu, Systems Sci. Math. Sci., 2(1989) 32–39.

Set

$$X^- = \overline{\bigoplus_{i\ge m} \ker(-\Delta - \lambda_i^\infty p_\infty)}, \qquad X^+ = \bigoplus_{i<m} \ker(-\Delta - \lambda_i^\infty p_\infty).$$

$\beta \succeq \lambda_{m-1}^\infty p_\infty$ implies

$$-\langle \nabla f(v+w_1) - \nabla f(v+w_2), w_1 - w_2\rangle \ge \kappa \|w_1 - w_2\|.$$

Applying Pro2, we obtain a reduced $\varphi \in C^1(X^-, \mathbb{R})$. Coercive?

Pob-2. For $\|v_n\| \to \infty$, since $\dim X^- = \infty$, the weak limit of $\|v_n\|^{-1} v_n$ may be the zero element in X^-, (P_∞^-) not apply.

As [Liu08], we consider $f_1 = f|_{X^-}$. Then $f_1 \in C^1(X^-, \mathbb{R})$.

Lem-2 (NVL). Let $\{v_n\} \subset X^-$ such that $f_1(v_n) \le c$ and $\|v_n\| \to \infty$. Set $v_n^0 = \|v_n\|^{-1} v_n$. Then up to sub, $v_n^0 \rightharpoonup v^0 \ne \mathbf{0}$.

Pf (In [Liu08], $\nabla f_1(v_n) \to 0$). Up to sub, $v_n^0 \rightharpoonup v^0$ in X^-, and

$$v_n^0 \to v^0 \qquad \text{in } L^2(\Omega).$$

[Liu08] S. Liu, Proc. Roy. Soc. Edinburgh Sect. A, 138(2008) 1281–1289.

By (4), $|2P(x,t)| \le \Lambda |t|^2$,

$$2c \ge 2f_1(v_n) = \int_\Omega |\nabla v_n|^2 \, dx - \int_\Omega 2P(x, v_n) dx.$$

$$\ge \int_\Omega |\nabla v_n|^2 \, dx - \Lambda \int_\Omega |v_n|^2 \, dx = \|v_n\|^2 - \Lambda |v_n|_2^2 .$$

Div by $\|v_n\|^2$ yields $2c \|v_n\|^{-2} \ge 1 - \Lambda \left| v_n^0 \right|_2^2$. we get $|v^0|_2^2 \ge \Lambda^{-1}$.

Lem-3. f_1 is coercive and bounded from below.

Pf. Assume for contradiction

$$f_1(v_n) \le c, \qquad \|v_n\| \to \infty. \tag{6}$$

Let $v_n^0 = \|v_n\|^{-1} v_n$, by Lem 2, up to sub $v_n^0 \rightharpoonup v^0 \ne \mathbf{0}$. Let

$$\Theta = \left\{ x \in \Omega | \; v^0(x) \ne 0 \right\},$$

then $|\Theta| > 0$. For $x \in \Theta$ we have $|v_n(x)| = \|v_n\| \left| v_n^0(x) \right| \to \infty$.

By (P_∞^-) and Fatou,

$$\int_\Theta \left(\frac{1}{2} p_\infty(x) v_n^2 - P(x, v_n) \right) dx \to +\infty, \qquad \text{as } n \to \infty.$$

(P_∞^-) implies $\exists M > 0$ such that

$$\frac{1}{2} p_\infty(x) t^2 - P(x, t) \geq -M, \qquad (x, t) \in \Omega \times \mathbb{R}.$$

Therefore $f_1(v_n) = \frac{1}{2} \int_\Omega |\nabla v_n|^2 \, dx - \int_\Omega P(x, v_n) dx$

$$\geq \int_\Omega \left(\frac{1}{2} p_\infty(x) v_n^2 - P(x, v_n) \right) dx.$$

$$= \left(\int_\Theta + \int_{\Omega \setminus \Theta} \right) \left(\frac{1}{2} p_\infty(x) v_n^2 - P(x, v_n) \right) dx$$

$$\geq \int_\Theta \left(\frac{1}{2} p_\infty(x) v_n^2 - P(x, v_n) \right) dx - M \, |\Omega \setminus \Theta| \to +\infty.$$

Rek-5. In [Liu08], old NVL (Lem 2) is used to show f_1 (PS), then obtain coerciveness of f_1 via [Li86].
The new approach does not use derivative information and much simpler..

Lem-4. In Thm 12, φ is coercive, bounded from below, (PS)..

Pf. From the coerciveness of f_1 and

$$\varphi(v) = \max_{w \in X^+} f(v+w) \geq f(v) = f_1(v),$$

φ is also coercive and b.f.b.. In particular, any (PS) sequence of φ is bounded. By Pro 3, $\nabla\varphi = \mathbf{1} -$ comp. So φ satisfies (PS).

[Liu08] S. Liu, Proc. Roy. Soc. Edinburgh Sect. A, 138(2008) 1281–1289.
[Li86] S. Li, Tech. Rep. IC/86/90, ICTP, Trieste, 1986.

Pf of Thm12. We prove the case (i). By Lem4, φ satisfies the (PS) condition, and bounded from below. Note that

$$\ell = \dim X^+ = d^\infty_{m-1},$$

by Thm3 and Lem1 we obtain

$$C_{d^0_k - d^\infty_{m-1}}(\varphi, \mathbf{0}) \cong C_{d^0_k}(f, \mathbf{0}) \neq 0. \tag{7}$$

Now, if $d^0_k \neq d^\infty_{m-1}$, the result follows from Thm4.

Rek-6. (1) Seems to be the first real application of Thm4.

(2) In our paper, actually we study elliptic systems.

$$\begin{cases} -\Delta u = F_u(x,u,v), & \text{in } \Omega, \\ -\Delta v = F_v(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

Our results improve those of [FdP].

[FdP] M. F. Furtado, F. O. V. de Paiva, Bull. Aust. Math. Soc, (in press).

Thank you!

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